



***On the mortality problem for matrices of low
dimensions***

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Abstract

In this paper, we discuss the existence of an algorithm to decide if a given set of 2×2 matrices is mortal: a set $F = \{A_1, \dots, A_m\}$ of 2×2 matrices is said to be *mortal* if there exist an integer $k \geq 1$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$. We survey this problem and propose some new extensions: we prove the problem to be BSS-undecidable for real matrices and Turing-decidable for two rational matrices. We relate the problem for rational matrices to the entry equivalence problem, to the zero in the left upper corner problem and to the reachability problem for piecewise affine functions. Finally, we state some NP-completeness results.

Keywords: Mortality, Matrices, Decidability

Résumé

Dans ce papier, nous discutons l'existence d'un algorithme pour décider si un ensemble donné de matrices 2×2 est mortel: un ensemble $F = \{A_1, \dots, A_m\}$ de matrices 2×2 est dit *mortel* s'il existe un entier $k \geq 1$ et des entiers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ avec $A_{i_1} A_{i_2} \cdots A_{i_k} = 0$. Nous présentons une synthèse des résultats connus sur ce problème et présentons quelques extensions: nous prouvons que le problème est BSS-indécidable pour les matrices réelles et Turing-décidable pour les matrices rationnelles. Nous relierons le problème au problème de l'égalité des coefficients, au problème du zéro dans un coin et au problème de l'atteignabilité pour les fonctions affines par morceaux. Enfin, nous établissons des résultats de NP-complétude.

Mots-clés: Mortalité, Matrices, Décidabilité

On the mortality problem for matrices of low dimensions

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Abstract

In this paper, we discuss the existence of an algorithm to decide if a given set of 2×2 matrices is mortal: a set $F = \{A_1, \dots, A_m\}$ of 2×2 matrices is said to be *mortal* if there exist an integer $k \geq 1$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} A_{i_2} \dots A_{i_k} = 0$. We survey this problem and propose some new extensions: we prove the problem to be BSS-undecidable for real matrices and Turing-decidable for two rational matrices. We relate the problem for rational matrices to the entry equivalence problem, to the zero in the left upper corner problem and to the reachability problem for piecewise affine functions. Finally, we state some NP-completeness results.

1 Introduction

Several undecidability results are known about problems involving matrices [5, 13]. For example, given a finite set F of matrices with integer entries, it is undecidable whether the semi-group generated by M contains a matrix having a zero in the right upper corner [16], is free [10, 7], or contains the zero matrix [19]. These problems have been proved to be undecidable when restricted to 3×3 matrices. But for both of them the question of their decidability or undecidability when restricted to 2×2 matrices remains open [5].

In this paper, we focus on the decidability of the last problem. A set $F = \{A_1, \dots, A_m\}$ of $d \times d$ matrices is said to be *mortal* if there exist an integer $k \geq 1$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} A_{i_2} \dots A_{i_k} = 0$. Hence, we focus on the following problem:

Open problem 1 *Is the decision problem $Mort_{\mathbb{Q}}(2)$ decidable?*

- *Instance: a finite set F of 2×2 matrices with rational entries.*
- *Question: is F mortal?*

The decidability of problem $Mort_{\mathbb{Q}}(2)$ remains unknown despite a lot of interest (see [14, 15] for some references and discussions).

The question of the decidability of $Mort_{\mathbb{Q}}(2)$ was first mentioned as an open problem in [21] and was formulated as follows “ Find an algorithm, which given a finite set H of non-singular linear transformations of the complex plane and lines L

and M through the origin, determines whether some product from H maps L onto M ".

When $K \in \{\mathbb{R}, \mathbb{Q}\}$, the problem $Mort_K(d)$ (respectively $Mort_K(d, m)$) will denote the following decision problem:

- Instance: a finite set F of $d \times d$ matrices with entries in K (respectively a finite set F of $m \times d \times d$ matrices with entries in K).
- Question: is F mortal?

One of our motivation is the following: deciding the mortality problem is equivalent to deciding the controllability of switched linear systems: given a system of the form $x(t+1) = A(t, u)x(t)$, where for all t the set of possible values of $A(t, u)$ is a finite set F of $d \times d$ matrices, the question of mortality of F corresponds to the controllability (to the origin) of such a system. Cf. [3].

One other motivation is to understand the frontier between decidability and undecidability for discrete time dynamical systems: proving that $Mort_{\mathbb{Q}}(2)$ is decidable or undecidable would really clarify the situation for discrete time hybrid and dynamical systems: cf [11] and [8]. For example the reachability problem for piecewise affine dynamical systems has been proved undecidable for 2-dimensional systems, but is open and related to the mortality problem (see section 4.3) for 1-dimensional systems [11].

Note that, if $Mort_{\mathbb{Q}}(2)$ turned out to be undecidable, it would surely give a way, which would extend the results of [1, 11, 18, 23], to simulate a Turing machine by a dynamical system of low dimension since most of the undecidability results known up to this date rely on simulations of Turing machines.

2 Links between dimension and number of matrices

Paterson proved in [19] that the problem restricted to 3×3 matrices is not decidable:

Theorem 1 ([19]) *$Mort_{\mathbb{Q}}(3)$ is recursively unsolvable.*

More precisely, Paterson proved in [19] that, if the Post Correspondence Problem (PCP) is undecidable with p rules, then $Mort_{\mathbb{Q}}(3, 2p+2)$ is undecidable.

Using the Modified Post Correspondence Problem (MPCP) instead of PCP, we improve this result and obtain:

Proposition 1 *Suppose that the Post Correspondence Problem is undecidable with p rules. Then problem $Mort_{\mathbb{Q}}(3, p+2)$ is undecidable.*

Proof: The Post Correspondence Problem (PCP) is the following: "given a finite set of couples of words $\{ \langle U_i, V_i \rangle \mid i = 1 \dots p \}$, determine if there exist a sequence of indexes i_1, i_2, \dots, i_k in $\{1, 2, \dots, p\}$ with $U_{i_1}U_{i_2} \dots U_{i_k} = V_{i_1}V_{i_2} \dots V_{i_k}$ ". It is proved in [19] that, to any instance $\{ \langle U_i, V_i \rangle \mid i = 1 \dots p \}$ of PCP can be associated a finite set $F = \{S, T, W(U_j, V_j), W'(U_j, V_j) \text{ for } j = 1, \dots, n\}$ of rational matrices which satisfy:

1. F is mortal if and only if there exists i_1, i_2, \dots, i_k in $\{1, 2, \dots, p\}$ with

$$SW'(U_{i_1}, V_{i_1})W(U_{i_2}, V_{i_2}) \dots W(U_{i_k}, V_{i_k})T = 0$$

2. for all integers i_1, i_2, \dots, i_k in $\{1, 2, \dots, p\}$,

$$SW'(U_{i_1}, V_{i_1})W(U_{i_2}, V_{i_2}) \dots W(U_{i_k}, V_{i_k})T = 0$$

if and only if $U_{i_1}U_{i_2} \dots U_{i_k} = V_{i_1}V_{i_2} \dots V_{i_k}$.

We replace the Post Correspondence Problem (PCP) by the Modified Post Correspondence Problem (MPCP) ¹ [9]: “Given a finite set of couples of words $\{ \langle U_i, V_i \rangle \mid i = 1 \dots p \}$, determine if there exists a sequence of indexes i_2, \dots, i_k in $\{1, 2, \dots, p\}$ with $U_1 U_{i_2} \dots U_{i_k} = V_1 V_{i_1} \dots V_{i_k}$ ”. Since PCP can be solved by p calls to MPCP, MPCP is undecidable with p rules when PCP is.

There remains only to prove that MPCP with p rules reduces to $Mort_{\mathbb{Q}}(3, p+2)$. Observing that in MPCP the first index i_1 must be 1, the set of matrices $F = \{T, SW'_{U_1, V_1}, W_{U_j, V_j} \text{ for } j = 1, \dots, n\}$ is mortal if and only if there exist i_2, \dots, i_k in $\{1, 2, \dots, p\}$ with $SW'(U_1, V_1)W(U_{i_2}, V_{i_2}) \dots W(U_{i_k}, V_{i_k})T = 0$, which in turns holds if and only if $\{ \langle U_i, V_i \rangle \mid i = 1 \dots p \}$ is a positive instance of MPCP. \square

Now, the following result is proved in [2] and in [5]:

Lemma 1 [2, 5] *For all $n \geq 2, m \geq 1$, $Mort_{\mathbb{Q}}(d, m)$ undecidable implies $Mort_{\mathbb{Q}}(dm, 2)$ undecidable.*

The minimal number p of rules for which PCP is undecidable is not known, but p is an integer between 3 and 7: see [17].

We obtain:

Corollary 1 *The following problems are undecidable:*

- $Mort_{\mathbb{Q}}(3, 9)$
- $Mort_{\mathbb{Q}}(27, 2)$

3 On the decidability of $Mort(2, 2)$.

We come back now to the decidability of the mortality problem for 2-dimensional matrices. We prove first that $Mort_{\mathbb{R}}(2, 2)$ is BSS-undecidable. Then we prove that $Mort_{\mathbb{Q}}(2, 2)$ is Turing-decidable.

We will use several times the following lemma:

Lemma 2 *A finite set $F = \{A_1, \dots, A_m\}$ of 2×2 matrices is mortal if and only if there exist an integer k and integers $i_1, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} \dots A_{i_k} = 0$, and*

1. $rank(A_{i_j}) = 2$ for $1 < j < k$,
2. $rank(A_{i_j}) < 2$ for $j \in \{1, k\}$.

Proof: Only the direct sense requires a proof. Suppose F is mortal. There exists a null product $A_{i_1} \dots A_{i_k} = 0$ with k minimal. We may assume $k \geq 2$, since otherwise the assertion is immediate. The matrices A_{i_1} and A_{i_k} of this product are non-singular because otherwise a null-product with few matrices could be obtained by multiplying $A_{i_1} \dots A_{i_k}$ by their inverse(s).

Let $j \geq 1$ be the smallest integer with $rank(A_{i_j}) < 2$. From $A_{i_1} \dots A_{i_k} = 0$, it follows that matrix $A_{i_1} \dots A_{i_{j-1}}$ sends the image I of matrix $A_{i_j} \dots A_{i_k}$ to 0. Now, I is also equal the image of A_{i_j} and is of dimension 1. Indeed, firstly, I is clearly included in the image of A_{i_j} , secondly, by definition of k , I can not be of dimension 0, and thirdly, the dimension of the image of A_{i_j} is at most 1 because $rank(A_{i_j}) < 2$. Hence $A_{i_1} \dots A_{i_{j-1}} A_{i_j} = 0$, which implies $j = k$, which in turns implies the direct sense of the lemma. \square

¹The difference between PCP and MPCP is that in the latter the first index i_1 must be equal to 1

3.1 BSS-undecidability of $Mort_{\mathbb{R}}(2, 2)$

Talking about the decidability or undecidability of $Mort_{\mathbb{R}}(2)$ requires to talk about machines that manipulate real-numbers.

One first approach would be consider the problem as a recursive analysis problem. However, this approach is non-appropriate because one can not decide whether a real number is equal to zero in this model [24].

A more natural approach is to consider the problem as a decidability problem for the real Turing machine proposed by Blum Shub and Smale in [4]. We assume that our reader knows this model: see [4] otherwise. Roughly speaking, a BSS-machine is a Random Access Machine [9] extended with real numbers: it has an unbounded number of real registers x_1, \dots, x_n, \dots which can contain any real number in unbounded precision and a finite number of built-in constants $\lambda_1, \dots, \lambda_m$. Its program is made of arithmetic operations between its real registers of type $x_i := x_j \# x_k$, for $\# \in \{+, -, *, /\}$, or of type $x_i := \lambda_j$, or of tests of type $x_i \# x_j$ with $\# \in \{>, \geq, <, \leq, =, \neq\}$. Let $\mathbb{R}^\infty = \cup_{i \in \mathbb{N}} \mathbb{R}^i$. An input $x \in \mathbb{R}^\infty$ (i.e. of type $x = (x_1, \dots, x_i, 0, \dots, 0, \dots)$ for some i) is said to be accepted by the machine if the program of the machine eventually halts when started with its real registers set to x . A language $L \subset \mathbb{R}^\infty$ is said to be *BSS-recursively enumerable* if it consists of the accepted inputs of some BSS-machine. The language L is said to be *BSS-recursive* if, in addition, its complement is BSS-recursively enumerable.

In other words, BSS recursive sets are those that can be decided using only arithmetical operations and tests. The reader should refer to [4] for more formal descriptions.

We recall a lemma proved in [4]: a set $S \subset \mathbb{R}^n$ for some n is said to be a *basic semi-algebraic set* if there exists a finite number of polynomials of n variables $p_1, p_2, \dots, p_{n_1}, p'_1, \dots, p'_{n_2}$ with $S = \{(x_1, \dots, x_n) \mid p_1(x_1, \dots, x_n) > 0 \wedge \dots \wedge p_{n_1}(x_1, \dots, x_n) > 0 \wedge p'_1(x_1, \dots, x_n) = 0 \wedge \dots \wedge p'_{n_2}(x_1, \dots, x_n) = 0\}$. A *semi-algebraic set* is a finite union of basic semi-algebraic sets.

Lemma 3 *Let $L \subset \mathbb{R}^\infty$ be a BSS-recursively enumerable set. Then L is a denumerable union of semi-algebraic sets.*

Sketch of proof: Write $L = \cup_{t \in \mathbb{N}} Acc_t$, where Acc_t is the subset of the inputs that are accepted by the machine at time t . Check that each subset Acc_t is a semi-algebraic set: see [4] for the formal details. \square

The remaining arguments of this subsection are inspired from [12] (observe that their seems to be a close relation between mortality and stability [12]).

We start by the following preliminary result:

Lemma 4 *Let $a, b, \lambda \in \mathbb{R}$ be some real numbers with $a^2 + b^2 \neq 0, \lambda \neq 0$. Let θ be an argument of complex number $a + ib$. The couple of matrices $F(a, b, \lambda) = \{A_1, A_2\}$ with*

$$A_1 = \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \quad A_2 = \begin{pmatrix} -\lambda & 1 \\ 0 & 0 \end{pmatrix}$$

is mortal if and only if there exists an integer $n \in \mathbb{N}$ with $\lambda = \tan(n\theta)$.

Proof: By lemma 2, $F(a, b, \lambda)$ is mortal if and only if there exists an integer $n \in \mathbb{N}$ with $A_2 A_1^n A_2 = 0$. This is in true if and only if there exists a n th power of A_1 which sends the image of A_2 to its kernel. Since $Im(A_2) = \langle (1, 0) \rangle$, $Ker(A_2) = \langle (1, \lambda) \rangle$, and since A_1 is the composition of an homothety and a rotation of angle θ , this is true if and only if there exists an integer $n \in \mathbb{N}$ with $\lambda = \tan(n\theta)$. \square

The following observations are easy:

Lemma 5 *Let θ be a real number. Let $E(\theta)$ be the subset of \mathbb{R} defined by*

$$E(\theta) = \{\lambda \mid \text{there exists an integer } n \in \mathbb{N} \text{ with } \lambda = \tan(n\theta)\}$$

1. $E(\theta)$ is a dense subset of \mathbb{R} if and only if $\theta/\pi \notin \mathbb{Q}$.
2. There exist two rational number $a, b \in \mathbb{Q}$ such that any argument θ of complex number $a + ib$ satisfies $\theta/\pi \notin \mathbb{Q}$: take for example $a = 1$ and $b = 2$ (see lemma 6).
3. When $\theta/\pi \notin \mathbb{Q}$, the complement $E^c(\theta)$ of $E(\theta)$ in \mathbb{R} has an uncountable number of connected components: actually, every point of $E^c(\theta)$ is its own connected component.

We can now prove that $Mort_{\mathbb{R}}(2, 2)$ is BSS-undecidable:

Theorem 2 *$Mort_{\mathbb{R}}(2, 2)$ is BSS-recursively enumerable but is not BSS-recursive.*

Proof: Since building a BSS-machine that semi-recognizes $Mort_{\mathbb{R}}(2, 2)$ is easy, the problem is BSS-recursively enumerable.

Representing the matrices by their coefficients, the space of the instances of problem $Mort_{\mathbb{R}}(2, 2)$ is \mathbb{R}^8 . Denote by $Pos \subset \mathbb{R}^8$ (resp. by $Neg \subset \mathbb{R}^8$) the subset of the positive (resp. negative) instances of the problem. Using lemma 3, we only need to prove that Neg is not a countable union of semi-algebraic sets.

Let $a, b \in \mathbb{Q}$ with $a + ib = \rho e^{i\theta}$, $\theta/\pi \notin \mathbb{Q}$ like in lemma 5. Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^8$ be the function that sends $\lambda \in \mathbb{R}$ to the couple of matrices $F(a, b, \lambda)$. By definition of γ , the image Im_{γ} of γ is an algebraic subset of \mathbb{R}^8 and γ realizes an homeomorphism between \mathbb{R} and Im_{γ} . By lemma 4, $\gamma^{-1}(Pos) = E(\theta)$ and $\gamma^{-1}(Neg) = E^c(\theta)$. Since γ is an homeomorphism, $E^c(\theta)$ and $\gamma(E^c(\theta)) = Neg \cap Im_{\gamma}$ must have the same number of connected components: i.e., by remark 3 of lemma 5, they must have an uncountable number of connected components.

Suppose $Neg = \cup_{i \in \mathbb{N}} S_i$ where each S_i is a semi-algebraic subset. It follows $Neg \cap Im_{\gamma} = \cup_{i \in \mathbb{N}} (Im_{\gamma} \cap S_i)$. Each of the $(Im_{\gamma} \cap S_i)$ must be a semi-algebraic subset because it is the intersection between an algebraic set and a semi-algebraic set. Since a semi-algebraic set has a finite number of connected components, $Neg \cap Im_{\gamma}$ must have a countable number of connected components. This is impossible by the previous paragraph. \square

We get immediately:

Corollary 2 • *For $n \geq 2, m \geq 2$, the problem $Mort_{\mathbb{R}}(n, m)$ is BSS-recursively enumerable but not BSS-recursive.*

- *$Mort_{\mathbb{R}}(2)$ is BSS-recursively enumerable but not BSS-recursive.*

However, observe that it is easy to extract the following fact from the proofs of the next section:

Theorem 3 *Problem $Mort_{\mathbb{R}}(2, 2)$ restricted to matrices with real eigen-values is BSS-recursive.*

Let us discuss the result of theorem 2 and of corollary 2: deciding whether a set of matrices is mortal using only arithmetical operations is not possible. But it does not mean that the problem can not be decided by an algorithm which uses non-arithmetical operations or which uses arguments about the semi-ring K of the entries for $K \neq \mathbb{R}$.

Thus, using some number-theoretical arguments, we prove in the next subsection that the problem $Mort_{\mathbb{Q}}(2, 2)$ is Turing-decidable (that is, decidable in the classical sense).

3.2 Turing-decidability of $Mort_{\mathbb{Q}}(2, 2)$

The decidability of $Mort_{\mathbb{Q}}(2, 2)$ has already be claimed [5, 20]. But the proofs was either wrong or incomplete (for example, contradicting the previous section) [20] or missing [5]. We present here a full proof.

The previous section shows that $Mort_{\mathbb{Q}}(2, 2)$ require the use of some number-theoretical arguments. Actually, the arguments we will use are given by the following result extracted from [22]:

Lemma 6 ([22]) • *The following decision problem is decidable:*

- *Instance:*
 - * *a rational number $p \in [-1, 1]$*
 - * *a rational number $q \in [-1, 1]$*
- *Question: does there exist $\theta \in \mathbb{R}$ and an integer $n \in \mathbb{N}$ with $\cos(\theta) = p$ and $\cos(n\theta) = q$?*
- *When $p \notin \{0, 1/2\}$, there are at most a finite number of such n and those n can be computed effectively.*

Proof: Write $p = r/s, q = u/v$ where r, s, u, v are some integers with $\gcd(r, s) = \gcd(u, v) = 1$. The decidability of the problem when $p = 0$ or $p = 1/2$ is trivial. Suppose $p \notin \{0, 1/2\}$. $\cos(n\theta)$ is a polynomial in $\cos(\theta)$ with integer coefficients. If this polynomial is written $\cos(n\theta) = p_n(r/s)$, then $s^n p_n(r/s)$ is some integer c_n which satisfies

$$2rc_{n+1} - s^2 c_n = c_{n+2} \quad (1)$$

with $c_1 = r$ et $c_2 = 2r^2 - s^2$ (if we denote $a_n = \sin(nx)$ and $b_n = \cos(nx)$, this recurrence comes from $a_{n+1} = a_1 b_n + b_1 a_n$, $b_{n+1} = b_1 b_n - a_1 a_n$, etc).

Suppose that s is not a power of 2. Write $s = 2^a b$, $s' = 2^{a'} b'$ with $b > 1$, $b' \geq 1$ odd. We are searching an integer n such that $c_n / (2^{an} b^n) = u / (2^{a'n'} b'^n)$. We claim $\gcd(c_n, b^n) = 1$ for all $n \in \mathbb{N}$. Indeed, if some odd integer d divides simultaneously s et c_n , then, since $\gcd(r, s) = 1$, the assertion $d|c_{n-1}, d|c_{n-2}, \dots, d|c_2$ implies $d|r^2$, which in turns implies $d = 1$. As a consequence an integer n candidate must satisfy $b_1 = b^n$. There are at most a finite number of such n and those n are computable.

Suppose now that s is a power of 2. Write $s = 2^k, k > 1$ (remember that we supposed $r/s \neq 1/2$). Write every c_n as $c_n = 2^{\lambda_n} v_n$ where v_n is an odd integer. Recurrence (1) becomes

$$2^{\lambda_{n+1}+1} r v_{n+1} - 2^{\lambda_n+2k} v_n = 2^{\lambda_{n+2}} v_{n+2} \quad (2)$$

We prove first that there exists an integer n with $\lambda_n + 1 < 2k + \lambda_{n-1}$: if it was false, we would always have $\lambda_n + 1 \geq 2k + \lambda_{n-1}$, so that $\lambda_n + 1 \geq 2(n-1)k + \lambda_1$ would hold for all n . Since $|\cos(n\theta)| < 1$, we have $kn \geq \lambda_n$ which implies $kn \geq 2(n-1)k + \lambda_1 - 1$ for all $n \in \mathbb{N}$. Clearly this is impossible.

Let n_0 be the smallest integer such that $\lambda_{n_0+1} < 2k + \lambda_{n_0}$: n_0 can be computed effectively by testing this condition for increasing n . We have $\lambda_{n_0+2} = \lambda_{n_0+1} + 1$, $\lambda_{n_0+2} + 1 = \lambda_{n_0+1} + 2 < \lambda_{n_0+1} + 2k$, so that for all integer $h \geq 0$, $\lambda_{n_0+2+h} = \lambda_{n_0+1+h} + 1$ holds. Hence, for each positive integer h , we must have $\lambda_{n_0+h} = \lambda_{n_0} + h$.

Now return to the existence of an integer n with $\cos(n\theta) = u/v$. For $\cos(\theta)$ having denominator 2^k , v must be a power of 2. Suppose $v = 2^m$. It may happen that there exists a solution for $n \leq n_0$. For $n > n_0$, a solution $n = n_0 + h$ must satisfy $\cos((n_0 + h)\theta) = v_{n_0+h} 2^{\lambda_{n_0}+h} / 2^{k(n_0+h)} = u/2^m$, hence $k(n_0 + h) - \lambda_{n_0} - h = m$, or $h = (m + \lambda_{n_0} - kn_0) / (k - 1)$. That is, the only integer n candidate exceeding n_0 is $n_0 + (m + \lambda_{n_0} - kn_0) / (k - 1)$. Hence, there at most $n_0 + 2$ integers n candidate to satisfy $\cos(n\theta) = u/v$ and those candidates are computable. \square

We get:

Theorem 4 *Problem $Mort_{\mathbb{Q}}(2, 2)$ is decidable.*

Proof: Let $F = \{A_1, A_2\}$ be an instance of the problem. Suppose without loss of generality that the rank of A_2 is greater than the rank of A_1 . If A_1 is of rank 2, then the two matrices are non-singular and F is non mortal by lemma 2. If A_1 is of rank 0 then F is mortal. If the two matrices have rank 1, by lemma 2, it suffices to test whether one of the products $A_1^2, A_1A_2, A_2A_1, A_2^2$ is null.

There remains only the case where A_2 is non singular and A_1 is of rank 1. By lemma 2, F is mortal if and only if there exists an integer $n \in \mathbb{N}$ with

$$A_1 A_2^n A_1 = 0 \quad (3)$$

We want to check this relation algebraically using the Jordan forms of the matrices A_1 and A_2 . Write

$$A_1 = P_1^{-1} J_1 P_1, \quad A_2 = P_2^{-1} J_2 P_2$$

$$J_1 = \begin{pmatrix} \kappa & 0 \\ 0 & 0 \end{pmatrix},$$

and

$$J_2 = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

with P_1 and P_2 non-singular. Eigen value κ is rational since κ is equal to the trace of rational matrix A_1 . Eigen values λ and μ are the (possibly complex) roots of the characteristic polynomial of rational matrix A_2 .

Equation 3 becomes

$$P_1^{-1} J_1 P_1 P_2^{-1} J_2^n P_2 P_1^{-1} J_1 P_1 = 0$$

or, since P_1 is non-singular,

$$J_1 P J_2^n P^{-1} J_1 = 0$$

where

$$P = P_1 P_2^{-1} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$$

Now, after substituting P, P^{-1} and J , when J_2 is of the first form, the problem is equivalent to testing whether there exists an integer $n \in \mathbb{N}$ with $ps\lambda^n - qr\mu^n = 0$, and when J_2 is of the second form, the problem is equivalent to testing whether there exists an integer $n \in \mathbb{N}$ with $(ps - qr)\lambda^n - rpn = 0$.

Suppose A_2 of the second form. Eigen value λ is rational because λ is equal to half the trace of rational matrix A_2 . It follows that the coefficients κ, p, q, r, s are easily computable rational numbers. Testing whether there exists an integer n with $(ps - qr)\lambda^n - rpn = 0$ is easy: compute numerically an approximation up to $1/2$ of real roots of equation $(ps - qr)\lambda^x - rpx = 0$. It gives at most two integers n candidate to be solution and check if one of the candidates is solution.

Suppose A_2 of the first form. We want to test the existence of an integer n with $ps\lambda^n - qr\mu^n = 0$. Observe that $\lambda \neq 0, \mu \neq 0$ since A_2 is of rank 2. λ, μ and the coefficients p, q, r, s can be complex numbers but are computable elements of $\mathbb{Q}(\lambda)$: that is, they are of the form $a + \lambda b$ for some computable rational numbers $a, b \in \mathbb{Q}$. By computing in $\mathbb{Q}(\lambda)$, the cases $ps = 0$ or $qr = 0$ are trivial. Suppose now $ps \neq 0$ and $qr \neq 0$. The problem is equivalent to testing whether there exists an integer n with $(\lambda/\mu)^n = (pq)/(rs)$. We must have $|\lambda/\mu|^n = |pq|/|rs|$. When $|\lambda/\mu| \neq 1$, n must be equal to $|pq|/(|rs| \log|\lambda/\mu|)$ and it suffices to compute

numerically an approximation up to $1/2$ of this real quantity to get at most two integers n candidate to be solutions. When $|\lambda/\mu| = 1$ and λ et μ are real numbers, we have necessarily $\lambda = \mu$ or $\lambda = -\mu$. In both cases, by computing in $\mathbb{Q}(\lambda)$ the problem is trivial. When $|\lambda/\mu| = 1$ and $|pq|/|rs| \neq 1$ the problem has no solution.

There remains only the case where λ and μ are two conjugated complex roots and $(pq)/(rs)$ is a complex number of modulus 1. In that case λ is a complex number with rational real part because λ is a root of the characteristic polynomial of matrix A_2 with rational coefficients. Therefore, complex numbers λ/μ and $(pq)/(rs)$ of type $a + \lambda b$ with computable $a, b \in \mathbb{Q}$ must also have rational computable real parts. If θ denotes an argument of complex number λ/μ of modulus 1, an integer n solution must satisfy $\cos(n\theta) = r'$ where r' is the real part of $(pq)/(rs)$. When the real part p' of λ/μ is equal to $1/2$, $n \mapsto (\lambda/\mu)^n$ is a periodic sequence of period 6: it suffices to check $(\lambda/\mu)^n = (pq)/(rs)$ for $n = 0, 1, \dots, 5$. Case $p' = 0$ can be dealt similarly. Now, when $p' \notin \{0, 1/2\}$, by lemma 6 there are at most a finite number of integers n satisfying $\cos(n\theta) = r'$ and that integers are computable. It suffices to check if equation $(\lambda/\mu)^n = (pq)/(rs)$ holds for those integers. \square

We have just proved that $Mort_{\mathbb{Q}}(2, 2)$ is Turing-decidable. We do not know whether $Mort_{\mathbb{Q}}(2, 3)$ is decidable. So our knowledge of the decidability of $Mort_{\mathbb{Q}}(2)$ stops at the previous theorem. However our proof of the BSS-undecidability of the problem shows that the problem is more a number-theoretical problem than an simple computability problem.

In next section, we will show that $Mort_{\mathbb{Q}}(2)$ can be related to other open problems of the literature.

4 Relations to other problems of the literature

In this section we prove that $Mort_{\mathbb{Q}}(2)$ is equivalent to the equality of entry problem studied by [14], to the zero in the corner problem studied by [16, 5] and can be linked to the problems studied by [11].

When C is a matrix, $C_{i,j}$ will denote the j^{th} coefficient of the i^{th} line of C .

4.1 Equality of entries

Here is a variation of theorem 2 of [14] (unlike theorem 2 of [14], we do not suppose F to be made only of non-singular matrices):

Lemma 7 *Let F be a finite set of 2×2 matrices with rational entries. There exists an integer k and some integers i_1, \dots, i_k such that $A_{i_1} \dots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$ if and only if the finite set F' made of the matrices of F and of the matrix*

$$H = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$$

is mortal.

Proof: First observe that $HCH = 0$ holds if and only if $C_{2,1} = C_{2,2}$. That proves the direct sense.

Conversely, by lemma 2, if F is mortal there exist i_1, \dots, i_k with $A_{i_1} \dots A_{i_k} = 0$, $A_{i_j} \neq H$ for $1 < j < k$, and $\text{rank}(A_{i_j}) < 2$ for $j \in \{1, k\}$. If $A_{i_1} = A_{i_k} = H$ the remark of the previous paragraph implies that $C = A_{i_2} \dots A_{i_{k-1}}$ satisfies $C_{2,1} = C_{2,2}$. If $A_{i_1} \neq H$ and $A_{i_k} \neq H$ then $A_{i_1} \dots A_{i_k}$ is a product of matrices of F equal to the null-matrix and the null-matrix O satisfies $O_{2,1} = O_{2,2}$. Now, for the remaining cases, observe that equation $HC = 0$ (resp. $CH = 0$) implies $C_{2,1} = C_{2,2}$. \square

We can now extend a result of [14]:

Theorem 5 (Equality of entries) *Problem $Mort_{\mathbb{Q}}(2)$ is equivalent to the following decision problem:*

- *Instance: a finite set $F = \{A_1, \dots, A_m\}$ of 2×2 matrices with rational entries.*
- *Question: does there exist an integer k and some integers i_1, \dots, i_k such that $A_{i_1} \dots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$?*

and to the following decision problem:

- *Instance: a finite set $F = \{A_1, \dots, A_m\}$ of non-singular 2×2 matrices with rational entries.*
- *Question: does there exist an integer k and some integers i_1, \dots, i_k such that $A_{i_1} \dots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$?*

Proof: Clearly the second problem reduces to the first. The first problem reduces to the mortality problem for 2×2 matrices by lemma 7 and a reduction from the mortality problem for 2×2 matrices to the second problem is given in [14]. \square

4.2 Zero in the left upper corner

It is known that the problem of deciding whether the semi-group generated by a finite set of 3×3 non-singular matrices contains an element with a zero in the right upper corner is undecidable [5, 16]. However, the decidability of the problem for 2×2 matrices is left open [5].

Nevertheless, this problem can be related to the mortality problem by:

Theorem 6 (Zero in left upper corner) *Problem $Mort_{\mathbb{Q}}(2)$ is equivalent to the following decision problem:*

- *Instance: a finite set $F = \{A_1, \dots, A_m\}$ of 2×2 matrices with rational entries.*
- *Question: does there exist an integer k and some integers i_1, \dots, i_k such that $A_{i_1} \dots A_{i_k}$ is a matrix C satisfying $C_{1,1} = 0$?*

and to the following decision problem:

- *Instance: a finite set $F = \{A_1, \dots, A_m\}$ of non-singular 2×2 matrices with rational entries.*
- *Question: does there exist an integer k and some integers i_1, \dots, i_k such that $A_{i_1} \dots A_{i_k}$ is a matrix C satisfying $C_{1,1} = 0$?*

Proof: Denote

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

Observing that, for all matrix C , matrix $C' = PCP^{-1}$ satisfies $C'_{1,1} = 0$ if and only if $C_{2,1} = C_{2,2}$, the above problems are equivalent to the equivalent problems of theorem 5 by conjugations by matrix P . \square

4.3 Restriction to lower triangular matrices

It was proposed in [14] to restrict the previous problems to lower triangular matrices. Indeed [19] also proves that the equality of entry problem is undecidable for lower triangular 3×3 matrices with rational entries.

Problem $Mort_{\mathbb{Q}}(2)$ restricted to lower triangular matrices is trivially decidable [14]: a finite set F of lower triangular matrices is mortal if and only if there exist two matrices A, B in F with $A_{1,1} = 0$ and $B_{2,2} = 0$. The zero in the left upper corner problem when restricted to lower triangular matrices becomes also trivial.

However, the answer to the following question is not known:

Open problem 2 (Lower triangular matrices) *Is the following decision problem decidable?*

- *Instance: a finite set $F = \{A_1, \dots, A_m\}$ of non-singular lower-triangular 2×2 matrices with rational entries.*
- *Question: does there exist an integer k and some integers i_1, \dots, i_k such that $A_{i_1} \dots A_{i_k}$ is a matrix C satisfying $C_{2,1} = C_{2,2}$?*

We prove that this problem can be related to an non-deterministic version of the open problem mentioned in [11]:

Theorem 7 *Open problem 2 is equivalent to the decidability of the following decision problem:*

- *Instance: a finite set $F = \{f_1, \dots, f_m\}$ of non-constant rational affine functions of dimension 1 (i.e. a set of functions of type $f_i : x \mapsto a_i x + b_i$, $a_i, b_i \in \mathbb{Q}$, $a_i \neq 0$).*
- *Question: does there exists a composition $f_{i_1} \circ f_{i_2} \circ \dots \circ f_{i_k}$ of these functions that maps point 0 to point 1?*

Proof: Call this problem the *composition problem*. Suppose that a finite set $F = \{A_1, \dots, A_m\}$ of non-singular lower-triangular matrices is given. Without loss of generality, we can suppose $A_{2,2} = 1$ for each matrix $A \in F$. Indeed, each matrix $A \in F$ must satisfy $A_{2,2} \neq 0$ to be non-singular, and replacing each matrix A by matrix $A/A_{2,2}$ in F does not change the mortality of set F .

Problem 2 reduces to the instance $F' = \{f_1, \dots, f_m\}$ of the composition problem where $f_i : x \mapsto (A_i)_{1,1}x + (A_i)_{2,1}$: any product $C = A_{i_1} \dots A_{i_k}$ of lower-triangular matrices with $(A_{i_j})_{2,2} = 1$ satisfies $C_{2,2} = 1$ and $C_{2,1} = f_{i_1} \circ f_{i_2} \dots f_{i_k}(0)$.

Conversely the composition problem reduces to problem 2: when a finite set $F = \{f_1, \dots, f_m\}$ of non-constant affine rational functions is given, $f_i : x \mapsto a_i x + b_i$, it suffices to consider $F' = \{A_1, \dots, A_m\}$ with

$$A_i = \begin{pmatrix} a_i & 0 \\ b_i & 1 \end{pmatrix}$$

and to observe that any product $C = A_{i_1} \dots A_{i_k}$ of matrices of this form satisfies $C_{2,2} = 1$ and $C_{2,1} = f_{i_1} \circ f_{i_2} \dots f_{i_k}(0)$. \square

5 NP-completeness results

5.1 K -length mortality

A set $F = \{A_1, \dots, A_m\}$ of $d \times d$ matrices is said to be *K -length mortal* if there exist an integer $k \leq K$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ with $A_{i_1} A_{i_2} \dots A_{i_k} = 0$.

Theorem 8 *Given a set F of m 3×3 matrices with rational entries and an integer $K \leq 1 + m/2$ the decision problem “Is F K -length-mortal?” is NP-hard.*

Proof: Via the reduction of [19] (or the proof of proposition 1) and the NP-completeness of Bounded PCP [6]. \square

Observe that [2] proves that this result remains true whenever the matrices are assumed to have entries in $\{0, 1\}$.

5.2 Mortality without repetition

When repetitions of matrices are not allowed, the problem becomes clearly also decidable: a set $F = \{A_1, \dots, A_m\}$ of $d \times d$ matrices is said to be *mortal without repetition* if there exist integers $k \geq 1$ and some integers $i_1, i_2, \dots, i_k \in \{1, \dots, m\}$ such that $A_{i_1} A_{i_2} \dots A_{i_k} = 0$ and $i_{j_1} \neq i_{j_2}$ for all $j_1 \neq j_2$.

Theorem 9 *Given a finite set F of m 2×2 matrices, and an integer K , the decision problem “Is F K -length-mortal without repetition?” is NP-hard in the strong sense.*

The proof uses a reduction from subset product [6]. We restate this problem here:

Proposition 2 (Subset Product (Yao)) *Given a finite set A , a size $s(a) \in \mathbb{N}^+$ for each $a \in A$, and a positive integer B , the decision problem “Is there a subset $A' \subset A$ such that the product of the sizes of the elements in A' is exactly B ?” is NP-complete in the strong sense.*

Proof:[of Theorem 9] Given an instance of subset product with $|A| = n$, define $n + 3$ matrices as follows:

$$\begin{pmatrix} 1 & 0 \\ 0 & s(a) \end{pmatrix}, \text{ for } a \in A, \quad \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}.$$

Note that we have repeated the last matrix, since we are required to use it twice. Denote the last matrix by H . Check that for all 2×2 matrix A , $HAH = 0$ if and only if $A_{2,1} = A_{2,2}$. Hence, by lemma 2, this set of matrices is mortal without repetition with length $4 \leq k \leq n + 3$ steps if and only if subset product has a solution in $1 \leq k - 3 \leq n$ steps. \square

6 Thanks

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